CHAPTER 9

DIFFERENTIAL EQUATIONS

POINTS TO REMEMBER

- **Differential Equation :** Equation containing derivatives of a dependant variable with respect to an independent variable is called differential equation.
- **Order of a Differential Equation :** The order of a differential equation is defined to be the order of the highest order derivative occurring in the differential equation.
- **Degree of a Differential Equation :** Highest power of highest order derivative involved in the equation is called degree of differential equation where equation is a polynomial equation in differential coefficients.
- **Formation of a Differential Equation :** We differentiate the family of curves as many times as the number of arbitrary constant in the given family of curves. Now eliminate the arbitrary constants from these equations. After elimination the equation obtained is differential equation.
- **Solution of Differential Equation**

  (i) **Variable Separable Method**

  $$\frac{dy}{dx} = f(x, y)$$

  We separate the variables and get

  $$f(x)dx = g(y)dy$$

  Then $\int f(x)\,dx = \int g(y)\,dy + c$ is the required solutions.

  (ii) ***Homogenous Differential Equation :*** A differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous functions of the same degree in x and y *i.e.,* of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

For solving this type of equations we substitute $y = vx$ and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The equation can be solved by variable separable method.

(iii) ***Linear Differential Equation :*** An equation of the from $\frac{dy}{dx} + Py = Q$ where P and Q are constant or functions of x only is called a linear differential equation. For finding solution of this type of equations, we find integrating factor $(I.F.) = e^{\int P\,dx}$.

Solution is $y\,(I.F.) = \int Q.\,(I.F.)\,dx + c$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the order and degree of the following differential equations.

(i) $\frac{dy}{dx} + \cos y = 0.$

(ii) $\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} = 4.$

(iii) $\frac{d^4y}{dx^4} + \sin x = \left(\frac{d^2y}{dx^2}\right)^5.$

(iv) $\frac{d^5y}{dx^5} + \log\left(\frac{dy}{dx}\right) = 0.$

(v) $\sqrt{1 + \frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}.$

(vi) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}.$

(vii) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = \sin x.$

(viii) $\frac{dy}{dx} + \tan\left(\frac{dy}{dx}\right) = 0$

2. Write the general solution of following differential equations.

(i) $\dfrac{dy}{dx} = x^5 + x^2 - \dfrac{2}{x}.$

(ii) $(e^x + e^{-x})\, dy = (e^x - e^{-x})dx$

(iii) $\dfrac{dy}{dx} = x^3 + e^x + x^e.$

(iv) $\dfrac{dy}{dx} = 5^{x+y}.$

(v) $\dfrac{dy}{dx} = \dfrac{1 - \cos 2x}{1 + \cos 2y}.$

(vi) $\dfrac{dy}{dx} = \dfrac{1 - 2y}{3x + 1}.$

3. Write integrating factor of the following differential equations

(i) $\dfrac{dy}{dx} + y \cos x = \sin x$

(ii) $\dfrac{dy}{dx} + y \sec^2 x = \sec x + \tan x$

(iii) $x^2 \dfrac{dy}{dx} + y = x^4.$

(iv) $x \dfrac{dy}{dx} + y \log x = x + y$

(v) $x \dfrac{dy}{dx} - 3y = x^3$

(vi) $\dfrac{dy}{dx} + y \tan x = \sec x$

(vii) $\dfrac{dy}{dx} + \dfrac{1}{1 + x^2} y = \sin x$

4. Write order of the differential equation of the family of following curves

(i) $y = Ae^x + Be^{x + c}$

(ii) $Ay = Bx^2$

(iii) $(x - a)^2 + (y - b)^2 = 9$

(iv) $Ax + By^2 = Bx^2 - Ay$

(v) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0.$

(vi) $y = a \cos (x + b)$

(vii) $y = a + be^{x+c}$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

5. (i) Show that $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$

(ii) Show that $y = \sin(\sin x)$ is a solution of differential equation

$$\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0.$$

(iii) Show that $y = Ax + \frac{B}{x}$ is a solution of $\frac{x^2 d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

(iv) Show that $y = a \cos(\log x) + b \sin(\log x)$ is a solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

(v) Verify that $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ satisfies the differential equation :

$$\left(a^2 + x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

(vi) Find the differential equation of the family of curves

$y = e^x (A \cos x + B \sin x)$, where A and B are arbitrary constants.

(vii) Find the differential equation of an ellipse with major and minor axes $2a$ and $2b$ respectively.

(viii) Form the differential equation representing the family of curves $(y - b)^2 = 4(x - a)$.

6. Solve the following differential equations.

(i) $\frac{dy}{dx} + y\cot x = \sin 2x.$ (ii) $x\frac{dy}{dx} + 2y = x^2 \log x.$

(iii) $\frac{dx}{dy} + \frac{1}{x} . y = \cos x + \frac{\sin x}{x}, \quad x > 0.$

(iv) $\cos^3 x \frac{dy}{dx} + \cos x = \sin x.$

(v) $ydx + \left(x - y^3\right) dy = 0$

(vi) $ye^y dx = \left(y^3 + 2xe^y\right) dy$

7. Solve each of the following differential equations :

(i) $y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right).$

(ii) $\cos y \, dx + (1 + 2e^{-x}) \sin y \, dy = 0.$

(iii) $x\sqrt{1 - y^2}dy + y\sqrt{1 - x^2}dx = 0.$

(iv) $\sqrt{\left(1 - x^2\right)\left(1 - y^2\right)} \, dy + xy \, dx = 0.$

(v) $(xy^2 + x) \, dx + (yx^2 + y) \, dy = 0; \; y(0) = 1.$

(vi) $\frac{dy}{dx} = y \sin^3 x \cos^3 x + xy \, e^x.$

(vii) $\tan x \, \tan y \, dx + \sec^2 x \sec^2 y \, dy = 0$

8. Solve the following differential equations :

(i) $x^2 \, y \, dx - (x^3 + y^3) \, dy = 0.$

(ii) $x^2 \frac{dy}{dx} = x^2 + xy + y^2.$

(iii) $\left(x^2 - y^2\right) dx + 2xy \, dy = 0, \quad y(1) = 1.$

(iv) $\left(y \sin \frac{x}{y}\right) dx = \left(x \sin \frac{x}{y} - y\right) dy.$ (v) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$

(vi) $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$ (vii) $\frac{dy}{dx} = e^{x+y} + x^2 e^y.$

(viii) $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}.$

(ix) $\left(3xy + y^2\right) dx + \left(x^2 + xy\right) dy = 0$

9. (i) Form the differential equation of the family of circles touching y-axis at (0, 0).

(ii) Form the differential equation of family of parabolas having vertex at (0, 0) and axis along the (i) positive y-axis (ii) positive x-axis.

(iii) Form differential equation of family of circles passing through origin and whose centre lie on x-axis.

(iv) Form the differential equation of the family of circles in the first quadrant and touching the coordinate axes.

10. Show that the differential equation $\frac{dy}{dx} = \frac{x + 2y}{x - 2y}$ is homogeneous and solve it.

11. Show that the differential equation :

$(x^2 + 2xy - y^2)\ dx + (y^2 + 2xy - x^2)\ dy = 0$ is homogeneous and solve it.

12. Solve the following differential equations :

(i) $\frac{dy}{dx} - 2y = \cos 3x.$

(ii) $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$ if $y\left(\frac{\pi}{2}\right) = 1$

(iii) $3e^x \tan y\, dx + \left(1 - e^x\right) \sec^2 y\, dy = 0$

13. Solve the following differential equations :

(i) $(x^3 + y^3)\ dx = (x^2y + xy^2)dy.$

(ii) $x\ dy - y\ dx = \sqrt{x^2 + y^2}dx.$

(iii) $y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}dx$

$$- x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}dy = 0.$$

(iv) $x^2dy + y(x + y)\ dx = 0$ given that $y = 1$ when $x = 1.$

(v) $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$ if $y(e) = 0$

(vi) $(x^3 - 3xy^2)\ dx = (y^3 - 3x^2y)dy.$

(vii) $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$ given that $y = 0$ when $x = 1$

16. Solve the following differential equations :

(i) $\cos^2 x\frac{dy}{dx} = \tan x - y.$

(ii) $x\cos x\frac{dy}{dx} + y\left(x\sin x + \cos x\right) = 1.$

(iii) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$

(iv) $(y - \sin x)\ dx + \tan x\ dy = 0,\ y(0) = 0.$

LONG ANSWER TYPE QUESTIONS (6 MARKS EACH)

17. Solve the following differential equations :

(i) $(x\,dy - y\,dx)\, y \sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\, x \cos\left(\frac{y}{x}\right)$

(ii) $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$ given that $y = \frac{\pi}{4}$, when $x = 1$.

(iii) $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$ given that $y(0) = 0$.

ANSWERS

1.(i) order = 1, degree = 1 (ii) order = 2, degree = 1

(iii) order = 4, degree = 1 (iv) order = 5, degree is not defined.

(v) order = 2, degree = 2 (vi) order = 2, degree = 2

(vii) order = 3, degree = 2 (viii) order = 1, degree is not defined

2.(i) $y = \frac{x^6}{6} + \frac{x^3}{6} - 2\log|x| + c$ (ii) $y = \log_e\left|e^x + e^{-x}\right| + c$

(iii) $y = \frac{x^4}{4} + e^x + \frac{x^{e+1}}{e+1} + c.$ (iv) $5^x + 5^{-y} = c$

(v) $2(y - x) + \sin 2y + \sin 2x = c.$ (vi) $2\log|3x + 1| + 3\log|1 - 2y| = c.$

3.(i) $e^{\sin x}$ (ii) $e^{\tan x}$

(iii) $e^{-1/x}$ (iv) $e^{\frac{(\log x)^2}{2}}$

(v) $\frac{1}{x^3}$ (vi) $\sec x$

(vii) $e^{\tan^{-1} x}$

4.(i) 1 (ii) 1

(iii) 2 (iv) 1

(v) 1 (vi) 1

(vii) 2

5.(vi) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ (vii) $x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx}$

(viii) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

6.(i) $y \sin x = \frac{2\sin^3 x}{3} + c$ (ii) $y = \frac{x^2(4\log_e x - 1)}{16} + \frac{c}{x^2}$

(iii) $y = \sin x + \frac{c}{x}, \; x > 0$ (iv) $y = \tan x - 1 + ce^{-\tan x}$

(v) $xy = \frac{y^4}{4} + c$ (vi) $x = -y^2e^{-y} + cy^2$

7.(i) $cy = (x + 2)(1 - 2y)$ (ii) $(e^x + 2)\sec y = c$

(iii) $\sqrt{1 - x^2} + \sqrt{1 - y^2} = c$

(iv) $\frac{1}{2}\log\left|\frac{\sqrt{1 - y^2} - 1}{\sqrt{1 - y^2} + 1}\right| = \sqrt{1 - x^2} - \sqrt{1 - y^2} + c$

(v) $(x^2 + 1)(y^2 + 1) = 2$

(vi) $\log y = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + xe^x - e^x + c$

$$= \frac{1}{16}\left[\frac{\cos^3 2x}{3} - \cos 2x\right] + (x-1)e^x + c$$

(vii) $\log|\tan y| - \frac{\cos 2x}{y} = c$

8.(i) $\frac{-x^3}{3y^3} + \log|y| = c$

(ii) $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$

(iii) $x^2 + y^2 = 2x$

(iv) $y = ce^{\cos(x/y)}$ [**Hint :** Put $\frac{1}{x} = v$]

(v) $\sin\left(\frac{y}{x}\right) = cx$

(vi) $c\left(x^2 - y^2\right) = y$

(vii) $-e^{-y} = e^x + \frac{x^3}{3} + c$

(viii) $\sin^{-1} y = \sin^{-1} x + c$

(ix) $x\log\left(x^3 y\right) + y = cx$

9.(i) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

(ii) $2y = x\frac{dy}{dx}$, $y = 2x\frac{dy}{dx}$

(iii) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

(iv) $(x-y)^2(1+y')^2 = (x+yy')^2$

10. $\log\left|x^2 + xy + y^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3x}}\right) + c$

11. $\frac{x^3}{x^2+y^2} = \frac{c}{x}(x+y)$

12.(i) $y = \frac{3\sin 3x}{13} - \frac{2\cos 3x}{13} + ce^{2x}$ (ii) $y = \frac{2}{3}\sin^2 x + \frac{1}{3}\operatorname{cosec} x$

(iii) $\tan y = k\left(1 - e^x\right)^3$

13.(i) $-y = x\log\{c(x - y)\}$ (ii) $cx^2 = y + \sqrt{x^2 + y^2}$

(iii) $xy\cos\left(\frac{y}{x}\right) = c$ (iv) $3x^2 y = y + 2x$

(v) $y = -x\log(\log|x|),\ x \neq 0$ (vi) $c\left(x^2 + y^2\right) = \sqrt{x^2 - y^2}$.

(vii) $\cos\frac{y}{x} = \log|x| + 1$

16. (i) $y = \tan x - 1 + ce^{\tan^{-1} x}$ (ii) $y = \frac{\sin x}{x} + c\frac{\cos x}{x}$

(iii) $x + ye^{\frac{x}{y}} = c$ (iv) $2y = \sin x$

17. (i) $c\,xy = \sec\left(\frac{y}{x}\right)$

(ii) $(1 - e)^3 \tan y = (1 - e^x)^3$

(iii) $y = x^2$.