CHAPTER 7

INTEGRALS

POINTS TO REMEMBER

- Integration is the reverse process of Differentiation.
- Let $\frac{d}{dx}F(x) = f(x)$ then we write $\int f(x)dx = F(x) + c$.
- These integrals are called indefinite integrals and c is called constant of integration.
- From geometrical point of view an indefinite integral is collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along y-axis.

STANDARD FORMULAE

1. $$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log|x| + c & n = -1 \end{cases}$$

2. $$\int (ax + b)^n \, dx = \begin{cases} \frac{(ax + b)^{n+1}}{(n+1)\,a} + c & n \neq -1 \\ \frac{1}{a}\log|ax + b| + c & n = -1 \end{cases}$$

3. $\int \sin x \; dx = -\cos x + c.$ 4. $\int \cos x \; dx = \sin x + c.$

5. $\int \tan x \, . \, dx = -\log|\cos x| + c = \log|\sec x| + c.$

6. $\int \cot x \, dx = \log|\sin x| + c.$ 7. $\int \sec^2 x \, . \, dx = \tan x + c.$

8. $\int \text{cosec}^2 x \, . \, dx = -\cot x + c.$ 9. $\int \sec x \, . \tan x \, . \, dx = \sec x + c.$

10. $\int \text{cosec } x \cot x \, dx = -\text{ cosec } x + c.$

11. $\int \sec x \, dx = \log|\sec x + \tan x| + c.$

12. $\int \text{cosec } x \, dx = \log|\text{cosec } x - \cot x| + c.$

13. $\int e^x dx = e^x + c.$ 14. $\int a^x dx = \frac{a^x}{\log a} + c$

15. $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c, \, |x < 1|.$

16. $\int \frac{1}{1 + x^2} dx = \tan^{-1} x + c.$

17. $\int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1} x + c, \, |x| > 1.$

18. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + c.$

19. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c.$

20. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c.$

21. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c.$

22. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \log \left| x + \sqrt{a^2 + x^2} \right| + c.$

23. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c.$

24. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c.$

25. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log \left| x + \sqrt{a^2 + x^2} \right| + c.$

26. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + c.$

RULES OF INTEGRATION

1. $\int k.f(x)dx = k\int f(x)dx.$

2. $\int k\{f(x) \pm g(x)\}dx = k\int f(x)dx \pm k\int g(x)dx.$

INTEGRATION BY SUBSTITUTION

1. $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c.$

2. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c.$

3. $\int \frac{f'(x)}{[f(x)]^n}\,dx = \frac{(f(x))^{-n+1}}{-n+1} + c.$

INTEGRATION BY PARTS

$$\int f(x).\,g(x)\,dx = f(x).\left[\int g(x)\,dx\right] - \int f'(x).\left[\int g(x)\,dx\right]dx.$$

DEFINITE INTEGRALS

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) = \int f(x)\,dx.$$

DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\left(a + \overline{n-1}\,h\right)\right]$$

where $h = \frac{b-a}{h}$. or $\int_a^b f(x)\,dx = \lim_{h\to 0}\left[h\sum_{r=1}^{n} f(a+rh)\right]$

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

3. $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

4. (i) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx.$ (ii) $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx.$

5. $\int_{-a}^{a} f(x) = 0$; if $f(x)$ is odd function.

6. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if $f(x)$ is even function.

7. $\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

Evaluate the following integrals

1. $\int \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}\right)dx.$

2. $\int_{-1}^{1} e^{|x|}dx.$

3. $\int \frac{1}{1-\sin^2 x}dx.$

4. $\int \left(8^x + x^8 + \frac{8}{x} + \frac{x}{8}\right)dx.$

5. $\int_{-1}^{1} x^{99}\cos^4 x\,dx.$

6. $\int \frac{1}{x\log x\log(\log x)}dx.$

7. $\int_{0}^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx.$

8. $\int \left(e^{a\log x} + e^{x\log a}\right)dx.$

9. $\int \left(\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\right)dx.$

10. $\int_{-\frac{\pi}{2}}^{\pi/2} \sin^7 x\,dx.$

11. $\int \left(x^c + c^x\right)dx.$

12. $\frac{d}{dx}\left[\int f(x)dx\right].$

13. $\int \frac{1}{\sin^2 x \cos^2 x} dx.$

14. $\int \frac{1}{\sqrt{x}+\sqrt{x-1}} dx.$

15. $\int e^{-\log e^x} dx.$

16. $\int \frac{e^x}{a^x} dx.$

17. $\int 2^x e^x dx.$

18. $\int \frac{x}{\sqrt{x+1}} dx.$

19. $\int \frac{x}{(x+1)^2} dx.$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx.$

21. $\int \cos^2 \alpha \, dx.$

22. $\int \frac{1}{x \cos \alpha + 1} dx.$

23. $\int \sec x . \log(\sec x + \tan x) \, dx.$

24. $\int \frac{1}{\cos \alpha + x \sin \alpha} dx.$

25. $\int \cot x . \log \sin x \, dx.$

26. $\int \left(x - \frac{1}{2}\right)^3 dx.$

27. $\int \frac{1}{x(2+3\log x)} dx.$

28. $\int \frac{1-\sin x}{x+\cos x} dx.$

29. $\int \frac{1-\cos x}{\sin x} dx.$

30. $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x} dx.$

31. $\int \frac{(x+1)}{x}(x+\log x) dx.$

32. $\int \left(\sqrt{ax} - \frac{1}{\sqrt{ax}}\right)^2 dx.$

33. $\int_0^{\pi} |\cos x| dx.$

34. $\int_0^2 [x] dx$ where [ ] is greatest integer function.

35. $\int_0^{\sqrt{2}} [x^2] dx$ where [ ] is greatest integer function.

36. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$.

37. $\int_{-2}^{1} \frac{|x|}{x} dx$.

38. $\int_{-1}^{1} x|x| dx$.

39. If $\int_0^a \frac{1}{1+x^2} = \frac{\pi}{4}$, then what is value of a.

40. $\int_a^b f(x)dx + \int_b^a f(x)dx$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

41. (i) $\int \frac{x \text{ cosec } (\tan^{-1} x^2)}{1+x^4} dx$.

(ii) $\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$.

(iii) $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$.

(iv) $\int \frac{\cos(x+a)}{\cos(x-a)} dx$.

(v) $\int \cos x \cos 2x \cos 3x \, dx$.

(vi) $\int \cos^5 x \, dx$.

(vii) $\int \sin^2 x \cos^4 x \, dx$.

(viii) $\int \cot^3 x \text{ cosec}^4 x \, dx$.

(ix) $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$.

(x) $\int \frac{1}{\sqrt{\cos^3 x \cos(x+a)}} dx$.

(xi) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$.

(xii) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$.

42. Evaluate :

(i) $\int \frac{x}{x^4 + x^2 + 1}\, dx.$

*(ii) $\int \frac{1}{x\left[6(\log x)^2 + 7\log x + 2\right]}\, dx.$

(iii) $\int \frac{dx}{1 + x - x^2}.$

(iv) $\int \frac{1}{\sqrt{9 + 8x - x^2}}\, dx.$

(v) $\int \frac{1}{\sqrt{(x - a)(x - b)}}\, dx.$

(vi) $\int \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}}\, dx.$

(vii) $\int \frac{5x - 2}{3x^2 + 2x + 1}\, dx.$

(viii) $\int \frac{x^2}{x^2 + 6x + 12}\, dx.$

(ix) $\int \frac{x + 2}{\sqrt{4x - x^2}}\, dx.$

(x) $\int x\sqrt{1 + x - x^2}\, dx.$

(xi) $\int (3x - 2)\sqrt{x^2 + x + 1}\, dx.$

(xii) $\int \sqrt{\sec x + 1}\, dx.$

43. Evaluate :

(i) $\int \frac{dx}{x\left(x^7 + 1\right)}.$

(ii) $\int \frac{\sin x}{(1 + \cos x)(2 + 3\cos x)}\, dx.$

(iii) $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}\, d\theta.$

(iv) $\int \frac{x-1}{(x+1)(x-2)(x+3)}\,dx.$

(v) $\int \frac{x^2+x+2}{(x-2)(x-1)}\,dx.$

(vi) $\int \frac{(x^2+1)(x^2+2)}{(x^3+3)(x^2+4)}\,dx.$

(vii) $\int \frac{dx}{(2x+1)(x^2+4)}.$

(viii) $\int \frac{dx}{\sin x\,(1-2\cos x)}.$

(ix) $\int \frac{\sin x}{\sin 4x}\,dx.$

(x) $\int \frac{x^2-1}{x^4+x^2+1}\,dx.$

(xi) $\int \sqrt{\tan x}\;dx.$

(xii) $\int \frac{x^2+9}{x^4+81}\,dx.$

44. Evaluate :

(i) $\int x^5 \sin x^3 dx.$

(ii) $\int \sec^3 x\,dx.$

(iii) $\int e^{ax}\cos(bx+c)\,dx.$

(iv) $\int \sin^{-1}\frac{6x}{1+9x^2}\,dx.$

(v) $\int \cos\sqrt{x}\;dx.$

(vi) $\int x^3 \tan^{-1} x\;dx.$

(vii) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx.$

(viii) $\int e^x\left(\frac{x-1}{2x^2}\right)dx.$

(ix) $\int \sqrt{2ax-x^2}\,dx.$

(x) $\int e^x \frac{(x^2+1)}{(x+1)^2}\,dx.$

(xi) $\int e^x \frac{(2+\sin 2x)}{(1+\cos 2x)}\,dx.$

(xii) $\int \left\{\log(\log x)+\frac{1}{(\log x)^2}\right\}dx.$

(xiii) $\int (6x + 5)\sqrt{6 + x - x^2}\,dx.$

(xiv) $\int (x - 2)\sqrt{\frac{x + 3}{x - 3}}dx.$

(xv) $\int (2x - 5)\sqrt{x^2 - 4x + 3}\,dx.$

(xvi) $\int \sqrt{x^2 - 4x + 8}\,\,dx.$

45. Evaluate the following definite integrals :

(i) $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\,dx.$

(ii) $\int_0^{\frac{\pi}{2}} \cos 2x \log \sin x\,\,dx.$

(iii) $\int_0^1 x\sqrt{\frac{1 - x^2}{1 + x^2}}\,dx.$

(iv) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1 - x^2)^{3/2}}\,dx.$

(v) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx.$

(vi) $\int_1^2 \frac{5x^2}{x^2 + 4x + 3}\,dx.$

(vii) $\int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x}\,dx.$

46. Evaluate :

(i) $\int_1^3 \{|x - 1| + |x - 2| + |x - 3|\}\,dx.$

(ii) $\int_0^{\pi} \frac{x}{1 + \sin x}\,dx.$

(iii) $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx.$ (iv) $\int_0^{\frac{\pi}{2}} \log \sin x\ dx.$

(v) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx.$

(vi) $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 2x - x^3 & \text{when } -2 \le x < 1 \\ x^3 - 3x + 2 & \text{when } -1 \le x < 1 \\ 3x - 2 & \text{when } 1 \le x < 2. \end{cases}$

(vii) $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,dx.$

(viii) $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\,dx.$

47. Evaluate the following integrals

(i) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ (ii) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx.$

(iii) $\int_{-1}^{1} \log\left(\frac{1+\sin x}{1-\sin x}\right)dx.$ (iv) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx.$

(v) $\int_0^{\pi} \frac{x \tan x}{\sec x \ \text{cosec}\ x}\,dx.$ (iv) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx.$

48. $\int_0^1 [2x]dx$ where [ ] is greatest integer function.

49. $\int e^{\log x + \log \sin x} dx$.

50. $\int e^{\log(x+1) - \log x} dx$.

51. $\int \frac{\sin x}{\sin 2x} dx$.

52. $\int \sin x \sin 2x \, dx$.

53. $\int_{-\frac{\pi}{4}}^{\pi/4} |\sin x| dx$.

54. $\int_a^b f(x)dx + \int_b^a f(a+b-x)\, dx$.

55. $\int \frac{1}{\sec x + \tan x} dx$.

56. $\int \frac{\sin^2 x}{1+\cos x} dx$.

57. $\int \frac{1-\tan x}{1+\tan x} dx$.

58. $\int \frac{a^x + b^x}{c^x} dx$.

59. Evaluate

(i) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx, \; x \in [0, 1]$

(ii) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \, dx$

(iii) $\int \frac{\sqrt{x^2+1}\left[\log(x^2+1) - 2\log x\right]}{x^4} dx$

(iv) $\int \frac{x^2}{(x \sin x + \cos x)^2} dx$

(v) $\int \sin^{-1}\sqrt{\frac{x}{a+x}} \, dx$

(vi) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

(vii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin|x| - \cos|x|)\, dx$

(viii) $\int_{1}^{2} [x^2]\, dx$, where $[x]$ is greatest integer function

(ix) $\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

60. Evaluate the following integrals :

(i) $\int \frac{x^5 + 4}{x^5 - x}\, dx$.

(ii) $\int \frac{dx}{(x-1)(x^2+4)}\, dx$

(iii) $\int \frac{2x^3}{(x+1)(x-3)^2}\, dx$

(iv) $\int \frac{x^4}{x^4 - 16}\, dx$

(v) $\int_{0}^{\frac{\pi}{2}} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$.

(vi) $\int \frac{1}{x^4 + 1}\, dx$.

(vii) $\int_{0}^{\infty} \frac{x \tan^{-1} x}{(1 + x^2)^2}\, dx$.

61. Evaluate the following integrals as limit of sums :

(i) $\int_{2}^{4} (2x + 1)\, dx$.

(ii) $\int_{0}^{2} (x^2 + 3)\, dx$.

(iii) $\int_1^3 (3x^2 - 2x + 4)\,dx$.

(iv) $\int_0^4 (3x^2 + e^{2x})\,dx$.

(v) $\int_2^5 (x^2 + 3x)\,dx$.

62. Evaluate

(i) $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$

(ii) $\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

(iii) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

(iv) $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$.

63. $\int \frac{1}{\sin x + \sin 2x}\,dx$.

64. $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta}\,d\theta$.

65. $\int \sec^3 x\,dx$.

66. $\int e^{2x}\cos 3x\,dx$.

ANSWERS

1. $\frac{\pi}{2}x + c$.

2. $2e - 2$

3. $\tan x + c$.

4. $\frac{8^x}{\log 8} + \frac{x^9}{9} + 8\log|x| + \frac{x^2}{16} + c$.

5. 0

6. $\log|\log(\log x)| + c$

7. 0

8. $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$

9. tan x + c

10. 0

11. $\frac{x^{c+1}}{c+1}+\frac{c^x}{\log c}+c$

12. $f(x)$ + c

13. tan x – cot x + c

14. $\frac{2}{3}x^{3/2}-\frac{2}{3}(x-1)^{3/2}+c$

15. log $|x|$ + c

16. $\left(\frac{e}{a}\right)^x \Big/ \log(e/a)+c$

17. $\frac{2^x e^x}{\log(2e)}+c$

18. $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+c.$

19. $\log|x+1|+\frac{1}{x+1}+c.$

20. $2e^{\sqrt{x}}+c$

21. $x\cos^2\alpha + c$

22. $\frac{\log|x\cos\alpha+1|}{\cos\alpha}+c.$

23. $\frac{(\log|\sec x+\tan x|)^2}{2}+c$

24. $\frac{\log|\cos\alpha+x\sin\alpha|}{\sin\alpha}+c$

25. $\frac{(\log\sin x)^2}{2}+c$

26. $\frac{x^4}{4}+\frac{1}{2x^2}-\frac{3x^2}{2}+3\log|x|+c.$

27. $\frac{1}{3}\log|2+3\log x|+c.$

28. log $|x + \cos x|$ c

29. 2 log $|\sec x/2|$ + $c.$

30. $\frac{1}{e}\log|x^e+e^x|+c.$

31. $\frac{(x+\log x)^2}{2}+c$

32. $a\frac{x^2}{2}+\frac{\log|ax|}{a}-2x+c.$

33. 0

34. 1

35. $(\sqrt{2}-1)$

36. $\frac{b-a}{2}$

37. –1

38. 0

39. 1

40. 0

41. (i) $\frac{1}{2}\log\left[\operatorname{cosec}(\tan^{-1}x^2)-\frac{1}{x^2}\right]+c.$

(ii) $\frac{1}{2}\left(x^2-x\sqrt{x^2-1}\right)+\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|+c.$

(iii) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+c$

(iv) $x\cos 2a-\sin 2a\log|\sec(x-a)|+c.$

(v) $\frac{1}{48}[12x+6\sin 2x+3\sin 4x+2\sin 6x]+c.$

(vi) $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c.$

(vii) $\frac{1}{32}\left[2x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 4x-\frac{1}{6}\sin 6x\right]+c.$

(viii) $-\left(\frac{\cot^6 x}{6}+\frac{\cot^4 x}{4}\right)+c.$

(ix) $\frac{1}{(a^2-b^2)\sqrt{a^2\sin^2 x+b^2\cos^2 x}}+c.$

[**Hint :** put $a^2\sin^2 x+b^2\cos^2 x=t$]

(x) $-2 \operatorname{cosec} a\sqrt{\cos a - \tan x \,.\, \sin a} + c.$

[**Hint. :** Take $\sec^2 x$ as numerator]

(xi) $\tan x - \cot x - 3x + c.$

(xii) $\sin^{-1}(\sin x - \cos x) + c.$

42. (i) $\dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{2x^2 + 1}{\sqrt{3}}\right) + c.$ [**Hint :** put $x^2 = t$]

(ii) $\log\left|\dfrac{2\log x + 1}{3\log x + 2}\right| + C$ [**Hint :** put $\log x = t$]

(iii) $\dfrac{1}{\sqrt{5}} \log\left|\dfrac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}\right| + c$

(iv) $\sin^{-1}\left(\dfrac{x - 4}{5}\right) + c.$

(v) $2\log\left|\sqrt{x - a} + \sqrt{x - b}\right| + c$

(vi)

$$-\cos\alpha \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha . \log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + c$$

$$\left[\textbf{Hint} : \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}} = \frac{\sin(x - \alpha)}{\sin^2 x - \sin^2\alpha}\right]$$

(vii) $\dfrac{5}{6}\log\left|3x^2 + 2x + 1\right| + \dfrac{(-11)}{3\sqrt{2}} \tan^{-1}\left(\dfrac{3x + 1}{\sqrt{2}}\right) + c$

(viii) $x - 3\log\left|x^2 + 6x + 12\right| + 2\sqrt{3} \tan^{-1}\left(\dfrac{x + 3}{\sqrt{3}}\right) + c$

(ix) $-\sqrt{4x - x^2} + 4 \sin^{-1}\left(\dfrac{x - 2}{2}\right) + c$

(x) $\frac{-1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2}$

$$+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$$

(xi) $\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{7}{2}\left[\begin{array}{l}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} \\ +\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\end{array}\right]+c$

(xii) $-\log\left|\cos x+\frac{1}{2}+\sqrt{\cos^2 x+\cos x}\right|+c$

[**Hint :** Multiply and divide by $\sqrt{\sec x+1}$]

43. (i) $\frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right|+c$

(ii) $\log\left|\frac{1+\cos x}{2+3\cos x}\right|+c$

(iii) $\frac{-2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|1+\cos\theta|+c.$

(iv) $\frac{9}{10}\log|x+3|+\frac{4}{15}\log|x-2|-\frac{1}{6}|x+1|+c$

(v) $x+4\log\left|\frac{(x-2)^2}{x-1}\right|+c$

(vi) $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+c$

[**Hint :** put $x^2 = t$]

(vii) $\frac{2}{17}\log|2x+1|-\frac{1}{17}\log|x^2+4|+\frac{1}{34}\tan^{-1}\frac{x}{2}+c$

(viii) $-\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + c$

[**Hint :** Multiply N^r and D^r by sin x and put cos $x = t$]

(ix) $\frac{-1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c$

(x) $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

(xi) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\tan x}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right| + c$

(xii) $\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x^2-9}{3\sqrt{2}}\right) + c$

44. (i) $\frac{1}{3}\left[-x^3\cos x^3 + \sin x^3\right] + c$

(ii) $\frac{1}{2}\left[\sec x\tan x + \log|\sec x + \tan x|\right] + c$

[**Hint :** Write $\sec^3 x = \sec x \cdot \sec^2 x$ and take sec x as first function]

(iii) $\frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right] + c_1$

(iv) $2x\tan^{-1}3x - \frac{1}{3}\log|1+9x^2| + c$ [**Hint :** put $3x = \tan\theta$]

(v) $2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

(vi) $\left(\frac{x^4-1}{4}\right)\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c.$

(vii) $\frac{1}{2}e^{2x}\tan x + c.$ (viii) $\frac{e^x}{2x} + c.$

(ix) $\frac{x-a}{2}\sqrt{2ax - x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$

(x) $e^x\left(\frac{x-1}{x+1}\right) + c.$

(xi) $e^x \tan x + c.$

(xii) $x \log|\log x| - \frac{x}{\log x} + c.$ [**Hint :** put $\log x = t \Rightarrow x = e^t$]

(xiii) $-2\left(6 + x - x^2\right)^{3/2}$
$+ 8\left[\frac{2x-1}{4}\sqrt{6 + x - x^2} + \frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right)\right] + c$

(xiv) $\frac{1}{2}(x+2)\sqrt{x^2 - 9} - \frac{3}{2}\log\left|x + \sqrt{x^2 - 9}\right| + c$

(xv) $\frac{2}{3}\left(x^2 - 4x + 3\right)^{3/2} - \left(\frac{x-2}{2}\right)\sqrt{x^2 - 4x + 3}$
$+ \frac{1}{2}\log\left|x - 2 + \sqrt{x^2 - 4x + 3}\right| + c$

(xvi) $\left(\frac{x-2}{2}\right)\sqrt{x^2 - 4x + 8} + 2\log\left|(x-2) + \sqrt{x^2 - 4x + 8}\right| + c$

45. (i) $\frac{1}{20}\log 3.$ (ii) $-\frac{\pi}{4}$

(iii) $\frac{\pi}{4} - \frac{1}{2}$. [**Hint :** put $x^2 = t$] (iv) $\frac{\pi}{4} - \frac{1}{2}\log 2$.

(v) $\frac{\pi}{2}$.

(vi) $5 - 10\log\frac{15}{8} + \frac{25}{2}\log\left(\frac{6}{5}\right)$.

(vii) $\pi/2$. $\left[\textbf{Hint :} \left(\frac{x}{1+\cos x} + \frac{\sin x}{1+\cos x}\right) dx.\right]$

46. (i) 8. (ii) π.

(iii) $\frac{\pi}{8}\log 2$. (iv) $\frac{-\pi}{2}\log 2$.

(v) $\frac{1}{4}\pi^2$.

(vi) 95/12.

$$\left[\textbf{Hint :} \int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx\right]$$

(vii) $\frac{\pi^2}{16}$.

(viii) $\frac{\pi^2}{2ab}$. $\left[\textbf{Hint :} \text{Use} \int_0^a f(x) = \int_0^a f(a-x)\right]$

47. (i) $\frac{\pi}{12}$. (ii) $\frac{\pi}{2} - \log 2$.

(iii) 0. (iv) $\pi/2$.

(v) $\frac{\pi^2}{4}$ (vi) $a\pi$.

48. $\frac{1}{2}$

49. $-x \cos x + \sin x + c$.

50. $x + \log x + c$.

51. $\frac{1}{2}\log|\sec x + \tan x| + c$.

52. $-\frac{1}{2}\left(\frac{\sin 3x}{3} - \sin x\right)$

53. $2 - \sqrt{2}$

54. 0

55. $\log|1 + \sin x| + c$

56. $x - \sin x + c$

57. $\log|\cos x + \sin x| + c$

58. $\frac{(a/c)^x}{\log(a/c)} + \frac{(b/c)^x}{\log(b/c)} + C$.

59. (i) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x - x^2}}{\pi} - x + c$

(ii) $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + c$

(iii) $-\frac{1}{3}\left(1 + \frac{1}{x^2}\right)^{3/2}\left[\log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3}\right] + c$

(iv) $\dfrac{\sin x - x\cos x}{x\sin x + \cos x} + c$

(v) $(x + a)\tan^{-1}\sqrt{\dfrac{x}{a}} - \sqrt{ax} + c$

(vi) $2\sin^{-1}\dfrac{\sqrt{3} - 1}{2}$

(vii) 0

(viii) $-\sqrt{2} - \sqrt{3} + 5$

(ix) $\dfrac{3}{\pi} + \dfrac{1}{\pi^2}$.

60. (i) $x - 4\log|x| + \dfrac{5}{4}\log|x - 1| + \dfrac{3}{4}\log|x + 1| + \log|x^2 + 1| - \dfrac{1}{2}\tan^{-1}x + c.$

$$x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + \log\left|\frac{x^2-1}{x^4+1}\right| + c.$$

(ii) $\dfrac{1}{5}\log|x - 1| - \dfrac{1}{10}\log|x^2 + 4| - \dfrac{1}{10}\tan^{-1}\left(\dfrac{x}{2}\right) + c.$

(iii) $2x - \dfrac{1}{8}\log|x + 1| + \dfrac{81}{8}\log|x - 3| - \dfrac{27}{2(x - 3)} + c.$

(iv) $x + \dfrac{1}{2}\log\left|\dfrac{x-2}{x+2}\right| - \tan^{-1}\left(\dfrac{x}{2}\right) + c.$

(v) $\pi/\sqrt{2}$.

(vi) $\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{(x^2 - 1)}{\sqrt{2x}} - \dfrac{1}{4\sqrt{2}}\log\left|\dfrac{x^2 - \sqrt{2x} + 1}{x^2 + \sqrt{2x} + 1}\right| + c$

(vii) $\pi/8$.

61. (i) 14. (ii) $\frac{26}{3}$.

(iii) 26.

(iv) $\frac{1}{2}\left(127 + e^8\right)$.

(v) $\frac{141}{2}$.

62. (i) $\frac{\pi}{2} - \log 2$

(ii) $-\frac{1}{5}\log\left|\frac{\tan x - x}{2\tan x + 1}\right| + c$

(iii) $\frac{\pi}{8}\log 2$.

(iv) $\frac{\pi}{2}\log\left(\frac{1}{2}\right)$.

63. $\frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log|1+2\cos x| + c$.

64. $3\log|(2-\sin\theta)| + \frac{4}{2-\sin\theta} + c$.

65. $\frac{1}{2}\sec x + \tan x + \frac{1}{2}\log|\sec x + \tan x| + c$.

66. $\frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c$.