CHAPTER 6

APPLICATIONS OF DERIVATIVES

POINTS TO REMEMBER

- **Rate of Change :** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left.\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$ represents the rate of change of y w.r.t. x at $x = x_0$.

- If $x = f(t)$ and $y = g(t)$

  By chain rule

  $$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \text{ if } \frac{dx}{dt} \neq 0.$$

- (i) A function $f(x)$ is said to be increasing (non-decreasing) on an interval (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2)$ $\forall$ $x_1, x_2 \in (a, b)$. Alternatively if $f'(x_0) \geq 0 \; \forall \; x \in (a, b)$, then $f(x)$ is increasing function in (a, b).

  (ii) A function $f(x)$ is said to be decreasing (non-increasing) on an interval (a, b). If $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \geq f(x_2)$ $\forall$ $x_1, x_2 \in (a, b)$. Alternatively if $f'(x) \leq 0$ $\forall$ $x \in (a, b)$, then $f(x)$ is decreasing function in (a, b).

- The equation of tangent at the point (x_0, y_0) to a curve $y = f(x)$ is given by

  $$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0).$$

where $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$ = slope of the tangent at the point (x_0, y_0).

(i) If $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$ does not exist then tangent is parallel to y-axis at (x_0, y_0) and its equation is $x = x_0$.

(ii) If tangent at $x = x_0$ is parallel to x-axis then $\left.\frac{dy}{dx}\right]_{x=x_0} = 0$

- Slope of the normal to the curve at the point (x_0, y_0) is given by $-\frac{1}{\left.\frac{dy}{dx}\right]_{x=x_0}}$.

- Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by

$$y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right]_{(x_0,y_0)}}(x - x_0).$$

- If $\left.\frac{dy}{dx}\right]_{(x_0,y_0)} = 0.$ then equation of the normal is $x = x_0$.

- If $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$ does not exit, then the normal is parallel to x-axis and the equation of the normal is $y = y_0$.

- Let $y = f(x)$

  Δx = the small increment in x and

  Δy be the increment in y corresponding to the increment in x

  Then approximate change in y is given by

  $dy = \left(\frac{dy}{dx}\right)\Delta x$ or $dy = f'(x)\,\Delta x$

  The approximate change in the value of f is given by

  $f(x + \Delta x) = f(x) + f'(x)\,\Delta x$

- Let f be a function. Let point c be in the domain of the function f at which either $f'(x) = 0$ or f is not derivable is called a critical point of f.

- **First Derivative Test :** Let f be a function defined on an open interval I. Let f be continuous at a critical point $c \in$ I. Then if,

  (i) $f'(x)$ changes sign from positive to negative as x increases through c, then c is called the point of the local maxima.

  (ii) $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of *local minima*.

  (iii) $f'(x)$ does not change sign as x increases through c, then c is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

- **Second Derivative Test :** Let f be a function defined on an interval I and let $c \in$ I. Then

  (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

  $f(c)$ is local maximum value of f.

  (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. $f(c)$ is local minimum value of f.

  (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of perimeter of the square.

2. The radius of the circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

3. If the radius of a soap bubble is increasing at the rate of $\frac{1}{2}$cm/sec. At what rate its volume is increasing when the radius is 1 cm.

4. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

5. The total revenue in rupees received from the sale of x units of a product is given by

   $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

6. Find the maximum and minimum values of function $f(x) = \sin 2x + 5$.

7. Find the maximum and minimum values (if any) of the function

   $$f(x) = -|x - 1| + 7 \quad \forall x \in R.$$

8. Find the value of a for which the function $f(x) = x^2 - 2ax + 6$, $x > 0$ is strictly increasing.

9. Write the interval for which the function $f(x) = \cos x$, $0 \le x \le 2\pi$ is decreasing.

10. What is the interval on which the function $f(x) = \frac{\log x}{x}$, $x \in (0, \infty)$ is increasing?

11. For which values of x, the functions $y = x^4 - \frac{4}{3}x^3$ is increasing?

12. Write the interval for which the function $f(x) = \frac{1}{x}$ is strictly decreasing.

13. Find the sub-interval of the interval $(0, \pi/2)$ in which the function $f(x) = \sin 3x$ is increasing.

14. Without using derivatives, find the maximum and minimum value of $y = |3 \sin x + 1|$.

15. If $f(x) = ax + \cos x$ is strictly increasing on R, find a.

16. Write the interval in which the function $f(x) = x^9 + 3x^7 + 64$ is increasing.

17. What is the slope of the tangent to the curve $f = x^3 - 5x + 3$ at the point whose x co-ordinate is 2?

18. At what point on the curve $y = x^2$ does the tangent make an angle of 45° with positive direction of the x-axis?

19. Find the point on the curve $y = 3x^2 - 12x + 9$ at which the tangent is parallel to x-axis.

20. What is the slope of the normal to the curve $y = 5x^2 - 4 \sin x$ at $x = 0$.

21. Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line with slope $-\frac{1}{6}$.

22. Find the point on the curve $y = x^2$ where the slope of the tangent is equal to the y – co-ordinate.

23. If the curves $y = 2e^x$ and $y = ae^{-x}$ intersect orthogonally (cut at right angles), what is the value of a?

24. Find the slope of the normal to the curve $y = 8x^2 - 3$ at $x = \frac{1}{4}$.

25. Find the rate of change of the total surface area of a cylinder of radius r and height h with respect to radius when height is equal to the radius of the base of cylinder.

26. Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing w.r.t. its radius when its radius is 3 cm?

27. For the curve $y = (2x + 1)^3$ find the rate of change of slope at $x = 1$.

28. Find the slope of the normal to the curve

$$x = 1 - a \sin \theta \quad ; \quad y = b \cos^2\theta \quad \text{at} \quad \theta = \frac{\pi}{2}$$

29. If a manufacturer's total cost function is $C(x) = 1000 + 40x + x^2$, where x is the out put, find the marginal cost for producing 20 units.

30. Find 'a' for which $f(x) = a(x + \sin x)$ is strictly increasing on R.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

31. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y co-ordinate is changing 8 times as fast as the x co-ordinate.

32. A ladder 5 metres long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 cm/sec. How fast is its height on the wall decreasing when the foot of the ladder is 4 metres away from the wall?

33. A balloon which always remain spherical is being inflated by pumping in 900 cubic cm of a gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

34. A man 2 meters high walks at a uniform speed of 5 km/hr away from a lamp post 6 metres high. Find the rate at which the length of his shadow increases.

35. Water is running out of a conical funnel at the rate of 5 cm^3/sec. If the radius of the base of the funnel is 10 cm and altitude is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the top.

36. The length x of a rectangle is decreasing at the rate of 5 cm/sec and the width y is increasing as the rate of 4 cm/sec when $x = 8$ cm and $y = 6$ cm. Find the rate of change of

    (a) Perimeter   (b) Area of the rectangle.

37. Sand is pouring from a pipe at the rate of 12c.c/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when height is 4 cm?

38. The area of an expanding rectangle is increasing at the rate of 48 cm^2/sec. The length of the rectangle is always equal to the square of the breadth. At what rate is the length increasing at the instant when the breadth is 4.5 cm?

39. Find a point on the curve $y = (x - 3)^2$ where the tangent is parallel to the line joining the points (4, 1) and (3, 0).

40. Find the equation of all lines having slope zero which are tangents to the curve $y = \dfrac{1}{x^2 - 2x + 3}$.

41. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

42. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

43. Show that the curves $4x = y^2$ and $4xy = k$ cut as right angles if $k^2 = 512$.

44. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - y + 5 = 0$.

45. Find the equation of the tangent to the curve $\sqrt{x}+\sqrt{y}=a$ at the point $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.

46. Find the points on the curve $4y = x^3$ where slope of the tangent is $\frac{16}{3}$.

47. Show that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

48. Find the equation of the tangent to the curve given by $x = a \sin^3 t$, $y = b \cos^3 t$ at a point where $t=\frac{\pi}{2}$.

49. Find the intervals in which the function $f(x) = \log(1 + x) - \frac{x}{1+x}, x > -1$ is increasing or decreasing.

50. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is

    (a) Increasing (b) Decreasing.

51. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in [0, 1].

52. Find the intervals on which the function $f(x)=\frac{x}{x^2+1}$ is decreasing.

53. Prove that $f(x)=\frac{x^3}{3}-x^2+9x,\ x\in[1,2]$ is strictly increasing. Hence find the minimum value of $f(x)$.

54. Find the intervals in which the function $f(x) = \sin^4 x + \cos^4 x,\ 0 \le x \le \frac{\pi}{2}$ is increasing or decreasing.

55. Find the least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

56. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}$, $x > 0$ is strictly decreasing.

57. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$, is strictly increasing on the interval $\left(0, \frac{\pi}{4}\right)$.

58. Show that the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$ is strictly increasing on $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$.

59. Show that the function $f(x) = \frac{\sin x}{x}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

Using differentials, find the approximate value of (Q. No. 60 to 64).

60. $(0.009)^{\frac{1}{3}}$.

61. $(255)^{\frac{1}{4}}$.

62. $(0.0037)^{\frac{1}{2}}$.

63. $\sqrt{0.037}$.

64. $\sqrt{25.3}$.

65. Find the approximate value of $f(5.001)$ where $f(x) = x^3 - 7x^2 + 15$.

66. Find the approximate value of $f(3.02)$ where $f(x) = 3x^2 + 5x + 3$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

67. Show that of all rectangles inscribed in a given fixed circle, the square has the maximum area.

68. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.

69. Show that of all the rectangles of given area, the square has the smallest perimeter.

70. Show that the right circular cone of least curved surface area and given volume has an altitude equal to $\sqrt{2}$ times the radium of the base.

71. Show that the semi vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

72. A point on the hypotenuse of a triangle is at a distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

73. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

74. Find the interval in which the function f given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

75. Find the intervals in which the function $f(x) = (x + 1)^3 (x - 3)^3$ is strictly increasing or strictly decreasing.

76. Find the local maximum and local minimum of $f(x) = \sin 2x - x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

77. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is strictly increasing or decreasing. Also find the points on which the tangents are parallel to x-axis.

78. A solid is formed by a cylinder of radius r and height h together with two hemisphere of radius r attached at each end. It the volume of the solid is constant but radius r is increasing at the rate of $\frac{1}{2\pi}$ metre/min. How fast must h (height) be changing when r and h are 10 metres.

79. Find the equation of the normal to the curve

    $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$ at the point θ and show that its distance from the origin is a.

80. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

81. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

82. Find the equation of the tangents at the points where the curve $2y = 3x^2 - 2x - 8$ cuts the x-axis and show that they make supplementary angles with the x-axis.

83. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

84. A window is in the form of a rectangle surmounted by an equilateral triangle. Given that the perimeter is 16 metres. Find the width of the window in order that the maximum amount of light may be admitted.

85. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet?

86. Find a point on the parabola $y^2 = 4x$ which is nearest to the point (2, –8).

87. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each cover and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum.

88. A window in the form of a rectangle is surmounted by a semi circular opening. The total perimeter of the window is 30 metres. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

89. An open box with square base is to be made out of a given iron sheet of area 27 sq. meter, show that the maximum value of the box is 13.5 cubic metres.

90. A wire of length 28 cm is to be cut into two pieces. One of the two pieces is to be made into a square and other in to a circle. What should be the length of two pieces so that the combined area of the square and the circle is minimum?

91. Show that the height of the cylinder of maximum volume which can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

92. Show that the altitude of the right circular cone of maximum volume that can be inscribed is a sphere of radius r is $\frac{4r}{3}$.

93. Prove that the surface area of solid cuboid of a square base and given volume is minimum, when it is a cube.

94. Show that the volume of the greatest cylinder which can be inscribed in a right circular cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

95. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

96. A given quantity of metal is to be cast half cylinder with a rectangular box and semicircular ends. Show that the total surface area is minimum when the ratio of the length of cylinder to the diameter of its semicircular ends is $\pi : (\pi + 2)$.

ANSWERS

1. 0.8 cm/sec.
2. 4.4 cm/sec.
3. 2π cm^3/sec.
4. 80π cm^2/sec.
5. Rs. 208.
6. Minimum value = 4, maximum value = 6.
7. Maximum value = 7, minimum value does not exist.
8. $a \le 0$.
9. $[0, \pi]$
10. $(0, e]$
11. $x \ge 1$
12. $(-\infty, 0) \cup (0, \infty)$
13. $\left(0, \frac{\pi}{6}\right)$.
14. Maximum value = 4, minimum valve = 0.
15. $a > 1$.
16. R
17. 7
18. $\left(\frac{1}{2}, \frac{1}{4}\right)$.
19. (2, – 3)
20. $\frac{1}{4}$
21. (1, 7)

22. (0, 0), (2, 4)

23. $\frac{1}{2}$.

24. $-\frac{1}{4}$.

25. $6\pi r$

26. $2\ \pi\ cm^2/cm$

27. 72

28. $-\frac{a}{2b}$.

29. Rs. 80.

30. $a > 0$.

31. $(4, 11)$ and $\left(-4, -\frac{31}{3}\right)$.

32. $-\frac{8}{3}$cm/sec.

33. $\frac{1}{\pi}$cm/sec.

34. 2.5 km/hr.

35. $\frac{4}{45\pi}$cm/sec.

36. (a) –2 cm/min, (b) 2 cm^2/min

37. $\frac{1}{48\pi}$cm/sec.

38. 7.11 cm/sec.

39. $\left(\frac{7}{2}, \frac{1}{4}\right)$.

40. $y = \frac{1}{2}$.

42. $2x + 3my = am^2\ (2 + 3m^2)$

44. $48x - 24y = 23$

45. $2x + 2y = a^2$

46. $\left(\frac{8}{3}, \frac{128}{27}\right), \left(\frac{-8}{3}, -\frac{128}{27}\right)$.

48. $y = 0$

49. Increasing in $(0, \infty)$, decreasing in $(-1, 0)$.

50. Increasing in $(-\infty, 2) \cup (6, \infty)$, Decreasing in $(2, 6)$.

52. (– ∞, –1) and (1, ∞).

53. $\frac{25}{3}$.

54. Increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ Decreasing in $\left(0, \frac{\pi}{4}\right)$.

55. $a = -2$.

56. Strictly decreasing in (1, ∞).

60. 0.2083

61. 3.9961

62. 0.06083

63. 0.1925

64. 5.03

65. –34.995

66. 45.46

68. 25, 10

74. Strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

Strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

75. Strictly increasing in (1, 3) ∪ (3, ∞)

Strictly decreasing in (–∞, –1) ∪ (–1, 1).

76. Local maxima at $x = \frac{\pi}{6}$

Local max. value $= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

Local minima at $x = -\frac{\pi}{6}$

Local minimum value $= \frac{-\sqrt{3}}{2} + \frac{\pi}{6}$

77. Strictly increasing in (–∞, 2] ∪ [3, ∞)

Strictly decreasing in (2, 3).

Points are (2, 29) and (3, 28).

78. $-\frac{3}{\pi}$ metres/min.

79. $x + y \tan\theta - a \sec\theta = 0$.

80. (0, 0), (–1, –2) and (1, 2).

81. $x + y = 3$

82. $5x - y - 10 = 0$ and $15x + 3y + 20 = 0$

83. $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1, \quad \frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0.$

84. $\frac{16}{6 - \sqrt{3}}$

85. $\sqrt{5}$

86. (4, –4)

87. 3cm

88. $\frac{60}{\pi + 4}, \frac{30}{\pi + 4}$.

90. $\frac{112}{\pi + 4}$ cm, $\frac{28\pi}{\pi + 4}$ cm.