CHAPTER 5

CONTINUITY AND DIFFERENTIATION

POINTS TO REMEMBER

- A function *f*(*x*) is said to be continuous at *x* = *c* iff $\lim_{x \to c} f(x) = f(c)$

  i.e., $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

- *f*(*x*) is continuous in (*a, b*) iff it is continuous at $x = c \; \forall \; c \in (a, b)$.

- *f*(*x*) is continuous in [*a, b*] iff

  (i) f(*x*) is continuous in (*a, b*)

  (ii) $\lim_{x \to a^+} f(x) = f(a)$,

  (iii) $\lim_{x \to b^-} f(x) = f(b)$

- Trigonometric functions are continuous in their respective domains.
- Every polynomial function is continuous on R.
- If *f* (*x*) and *g* (*x*) are two continuous functions and *c* ∈ *R* then at *x* = *a*

  (i) *f* (*x*) ± *g* (*x*) are also continuous functions at *x* = *a*.

  (ii) *g* (*x*) . *f* (*x*), *f* (*x*) + *c*, *cf* (*x*), | *f* (*x*)| are also continuous at *x* = *a*.

  (iii) $\frac{f(x)}{g(x)}$ is continuous at *x* = *a* provided *g*(*a*) ≠ 0.

- *f* (*x*) is derivable at *x* = *c* in its domain iff

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}, \quad \text{and is finite}$$

The value of above limit is denoted by $f'(c)$ and is called the derivative of $f(x)$ at $x = c$.

- $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

- If $y = f(u)$ and $u = g(t)$ then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u).g'(t)$ (Chain Rule)

- If $y = f(u)$, $x = g(u)$ then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}.$$

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$, $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$, $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$, $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$

$\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\log x) = \frac{1}{x}$

- $f(x) = [x]$ is discontinuous at all integral points and continuous for all $x \in R - Z$.

- **Rolle's theorem :** If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) and $f(a) = f(b)$ then there exists atleast one real number $c \in (a, b)$ such that $f'(c) = 0$.

- **Mean Value Theorem :** If $f(x)$ is continuous in $[a, b]$ and derivable in (a, b) then there exists atleast one real number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- $f(x) = \log_e x$, $(x > 0)$ is continuous function.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. For what value of x, $f(x) = |2x - 7|$ is not derivable.
2. Write the set of points of continuity of $g(x) = |x - 1| + |x + 1|$.
3. What is derivative of $|x - 3|$ at $x = -1$.
4. What are the points of discontinuity of $f(x) = \dfrac{(x-1)+(x+1)}{(x-7)(x-6)}$.
5. Write the number of points of discontinuity of $f(x) = [x]$ in $[3, 7]$.
6. The function, $f(x) = \begin{cases} \lambda x - 3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$ is a continuous function for all $x \in R$, find λ.
7. For what value of K, $f(x) = \begin{cases} \dfrac{\tan 3x}{\sin 2x}, & x \neq 0 \\ 2K, & x = 0 \end{cases}$ is continuous $\forall\ x \in R$.
8. Write derivative of $\sin x$ *w.r.t.* $\cos x$.
9. If $f(x) = x^2 g(x)$ and $g(1) = 6$, $g'(1) = 3$ find value of $f'(1)$.
10. Write the derivative of the following functions :

    (i) $\log_3 (3x + 5)$ (ii) $e^{\log_2 x}$

    (iii) $e^{6 \log_e (x-1)}, x > 1$

(iv) $\sec^{-1}\sqrt{x} + \mathrm{cosec}^{-1}\sqrt{x},\ x \geq 1.$

(v) $\sin^{-1}\left(x^{7/2}\right)$ (vi) $\log_x 5,\ x > 0.$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

11. Discuss the continuity of following functions at the indicated points.

(i) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x = 0.$

(ii) $g(x) = \begin{cases} \dfrac{\sin 2x}{3x}, & x \neq 0 \\ \dfrac{3}{2} & x = 0 \end{cases}$ at $x = 0.$

(iii) $f(x) = \begin{cases} x^2 \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0.$

(iv) $f(x) = |x| + |x - 1|$ at $x = 1.$

(v) $f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0 & x = 1 \end{cases}$ at $x = 1.$

12. For what value of k, $f(x) = \begin{bmatrix} 3x^2 - kx + 5, & 0 \leq x < 2 \\ 1 - 3x & 2 \leq x \leq 3 \end{bmatrix}$ is continuous $\forall\ x \in [0, 3].$

13. For what values of a and b

$$f(x) = \begin{cases} \dfrac{x + 2}{|x + 2|} + a & \text{if } x < -2 \\ a + b & \text{if } x = -2 \\ \dfrac{x + 2}{|x + 2|} + 2b & \text{if } x > -2 \end{cases}$$ is continuous at $x = 2.$

14. Prove that $f(x) = |x + 1|$ is continuous at $x = -1$, but not derivable at $x = -1$.

15. For what value of p,

$$f(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$ is derivable at $x = 0$.

16. If $y = \frac{1}{2}\left[\tan^{-1}\left(\frac{2x}{1-x^2}\right) + 2\tan^{-1}\left(\frac{1}{x}\right)\right]$, $0 < x < 1$, find $\frac{dy}{dx}$.

17. If $y = \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ then $\frac{dy}{dx} = ?$

18. If $5^x + 5^y = 5^{x+y}$ then prove that $\frac{dy}{dx} + 5^{y-x} = 0$.

19. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ then show that $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

20. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

21. If $(x + y)^{m+n} = x^m \cdot y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$.

22. Find the derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

23. Find the derivative of $\log_e(\sin x)$ w.r.t. $\log_a(\cos x)$.

24. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

25. If $x = a\cos^3\theta$, $y = a\sin^3\theta$ then find $\frac{d^2y}{dx^2}$.

26. If $x = ae^t$ (sint – cos t)

$y = ae^t$ (sint + cost) then show that $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is 1.

27. If $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$ then find $-\frac{dy}{dx}$.

28. If $y = x^{\log_e x} + (\log_e x)^x$ then find $\frac{dy}{dx}$.

29. Differentiate x^{x^x} w.r.t. x.

30. Find $\frac{dy}{dx}$, if $(\cos x)^y = (\cos y)^x$

31. If $y = \tan^{-1}\left(\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right)$ where $\frac{\pi}{2} < x < \pi$ find $\frac{dy}{dx}$.

32. If $x = \sin\left(\frac{1}{a}\log_e y\right)$ then show that $(1 - x^2)\, y'' - xy' - a^2 y = 0$.

33. Differentiate $(\log x)^{\log x}, x > 1$ *w.r.t.* x

34. If sin $y = x$ sin $(a + y)$ then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

35. If $y = \sin^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y.

36. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then show that $\frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}$.

37. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

38. If $y^3 = 3ax^2 - x^3$ then prove that $\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}$.

39. Verify Rolle's theorem for the function, $y = x^2 + 2$ in the interval $[a, b]$ where $a = -2$, $b = 2$.

40. Verify Mean Value Theorem for the function, $f(x) = x^2$ in [2, 4]

ANSWERS

1. $x = -7/2$

2. R

3. -1

4. $x = 6, 7$

5. Points of discontinuity of $f(x)$ are 4, 5, 6, 7 *i.e.* four points.

   Note : At $x = 3$, $f(x) = [x]$ is continuous. because $\lim_{x \to 3^+} f(x) = 3 = f(3)$.

6. $\lambda = \frac{7}{2}$.

7. $k = \frac{3}{4}$.

8. $-\cot x$

9. 15

10. (i) $\frac{3}{3x + 5} \log_3 e$

    (ii) $e^{\log_2 x} \frac{1}{x} . \log_2 e$.

    (iii) $6 (x - 1)^5$

    (iv) 0

    (v) $\frac{7}{2} \frac{x^2 \sqrt{x}}{\sqrt{1 - x^7}}$.

    (vi) $\frac{-\log_e 5}{x(\log_e x)^2}$.

11. (i) Discontinuous

    (ii) Discontinuous

    (iii) Continuous

    (iv) continuous

    (v) Discontinuous

12. k = 11

13. $a = 0$, $b = -1$.

15. $p > 1$.

16. 0

17. $\frac{-x}{\sqrt{1 - x^2}}$.

22. 1

23. $-\cot^2 x \log_e a$

24. $\frac{dy}{dx} = \frac{-x^x (1 + \log x) - yx^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$.

25. $\frac{d^2y}{dx^2} = \frac{1}{3a} \operatorname{cosec} \theta \sec^4 \theta.$

27. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}.$

28. $x^{\log x} \frac{2\log x}{x} + (\log x)^x \left[\frac{1}{\log x} + \log(\log x)\right].$

29. $\frac{dy}{dx} = x^{x^x} . x^x \log x \left(1 + \log x + \frac{1}{x \log x}\right).$

30. $\frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$

31. $\frac{dy}{dx} = -\frac{1}{2}.$

Hint. : $\sin\frac{x}{2} > \cos\frac{x}{2}$ *for* $x \in \left(\frac{\pi}{2}, \pi\right).$

33. $(\log x)^{\log x} \left[\frac{1}{x} + \frac{\log(\log x)}{x}\right], \; x > 1$

35. $\sec^2 y$ tany.