CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

IMPORTANT POINTS

- $\sin^{-1} x$, $\cos^{-1} x$, ... etc., are angles.
- If $\sin\theta = x$ and $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ then $\theta = \sin^{-1}x$ etc.
Function	*Domain*	*Range (Principal Value Branch)*
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1}x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sin^{-1}(\sin x) = x \ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  $\cos^{-1}(\cos x) = x \ \forall \ x \in [0, \pi]$ etc.
- $\sin(\sin^{-1}x) = x \ \forall \ x \in [-1, 1]$

  $\cos(\cos^{-1}x) = x \ \forall \ x \in [-1, 1]$ etc.

- $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)\ \forall x \in [-1, 1]$

  $\tan^{-1}x = \cot^{-1}(1/x)\ \forall\ x > 0$

  $\sec^{-1}x = \cos^{-1}(1/x),\ \forall\ |x| \geq 1$

- $\sin^{-1}(-x) = -\sin^{-1}x\ \forall\ x \in [-1, 1]$

  $\tan^{-1}(-x) = -\tan^{-1}x\ \forall\ x \in R$

  $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x\ \forall\ |x| \geq 1$

- $\cos^{-1}(-x) = \pi - \cos^{-1}x\ \forall\ x \in [-1, 1]$

  $\cot^{-1}(-x) = \pi - \cot^{-1}x\ \forall\ x \in -R$

  $\sec^{-1}(-x) = \pi - \sec^{-1}x\ \forall\ |x| \geq 1$

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},\quad x \in [-1, 1]$

  $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\ \forall\ x \in R$

  $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}\ \forall\ |x| \geq 1$

- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right);\quad xy < 1.$

- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right);\quad xy > -1.$

- $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right), |x| < 1$

  $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right), |x| \leq 1,$

  $2\tan^{-1}x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), x \geq 0.$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the principal value of

(i) $\sin^{-1}\left(-\sqrt{3}/2\right)$ (ii) $\cos^{-1}\left(\sqrt{3}/2\right)$.

(iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iv) $\text{cosec}^{-1}(-2)$.

(v) $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$. (vi) $\sec^{-1}(-2)$.

(vii) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\left(-1/\sqrt{3}\right)$

2. What is value of the following functions (using principal value).

(i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$. (ii) $\sin^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

(iii) $\tan^{-1}(1) - \cot^{-1}(-1)$. (iv) $\text{cosec}^{-1}\left(\sqrt{2}\right) + \sec^{-1}\left(\sqrt{2}\right)$.

(v) $\tan^{-1}(1) + \cot^{-1}(1) + \sin^{-1}(1)$.

(vi) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. (vii) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$.

(viii) $\text{cosec}^{-1}\left(\text{cosec}\,\frac{3\pi}{4}\right)$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

3. Show that $\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right) = \frac{\pi}{4} + \frac{x}{2}$. $x \in [0, \pi]$

4. Prove

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) - \cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right) = \frac{\pi}{4} \qquad x \in (0, \pi/2).$$

5. Prove $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a} = \cos^{-1}\left(\frac{\sqrt{a^2-x^2}}{a}\right).$

6. Prove

$$\cot^{-1}\left[2\tan\left(\cos^{-1}\frac{8}{17}\right)\right] + \tan^{-1}\left[2\tan\left(\sin^{-1}\frac{8}{17}\right)\right] = \tan^{-1}\left(\frac{300}{161}\right).$$

7. Prove $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2.$

8. Solve $\cot^{-1}2x + \cot^{-1}3x = \frac{\pi}{4}.$

9. Prove that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}, m, n > 0$

10. Prove that $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right] = \frac{x+y}{1-xy}$

11. Solve for x, $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \frac{1}{2}\tan^{-1}\left(\frac{-2x}{1-x^2}\right) = \frac{2\pi}{3}$

12. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

13. Solve for x, $\tan(\cos^{-1}x) = \sin(\tan^{-1}2);\ x > 0$

14. Prove that $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{32}{43}\right)$

15. Evaluate $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]$

16. Prove that $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x$

17. Prove that

$$\cot\left\{\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right\} + \cos^{-1}(1-2x^2) + \cos^{-1}(2x^2-1) = \pi, \quad x > 0$$

18. Prove that $\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right) = 0$ where a, b, $c > 0$

19. Solve for x, $2\tan^{-1}(\cos x) = \tan^{-1}(2\ \text{cosec}\ x)$

20. Express $\sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$ in simplest form.

21. If $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$, then prove that $a + b + c = abc$

22. If $\sin^{-1}x > \cos^{-1}x$, then x belongs to which interval?

ANSWERS

1. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{6}$ (iii) $\frac{-\pi}{6}$ (iv) $\frac{-\pi}{6}$

   (v) $\frac{\pi}{3}$ (vi) $\frac{2\pi}{3}$ (vii) $\frac{\pi}{6}$.

2. (i) 0 (ii) $\frac{-\pi}{3}$ (iii) $-\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$

   (v) π (vi) $\frac{\pi}{5}$ (vii) $\frac{-\pi}{6}$ (viii) $\frac{\pi}{4}$.

8. 1

11. $\tan\frac{\pi}{12} = 2 - \sqrt{3}$

13. $\frac{\sqrt{5}}{3}$

15. $\sqrt{\frac{\sqrt{11}-3}{3+\sqrt{11}}}$

19. $x = \frac{\pi}{4}$.

20 $\sin^{-1} x - \sin^{-1} \sqrt{x}$.

22. $\left(\frac{1}{\sqrt{2}}, 1\right]$

21. ***Hint:*** Let $\tan^{-1} a = \alpha$

$\tan^{-1} b = \beta$

$\tan^{-1} c = \gamma$

then given, $\alpha + \beta + \gamma = \pi$

$\therefore \quad \alpha + \beta = \pi - \gamma$

take tangent on both sides,

$\tan(\alpha + \beta) = \tan(\pi - \gamma)$