CHAPTER 11

THREE DIMENSIONAL GEOMETRY

POINTS TO REMEMBER

- Distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- (i) The coordinates of point R which divides line segment PQ where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ internally are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right).$$

  (ii) The co-ordinates of a point which divides join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio of $m : n$ externally are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right).$$

- Direction ratios of a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- Direction cosines of a line whose direction ratios are a, b, c are given by

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \; n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

- (i) Vector equation of a line through point $\vec{a}$ and parallel to vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

  (ii) Cartesian equation of a line through point (x_1, y_1, z_1) and having direction ratios proportional to a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

- (i) Vector equation of line through two points

  $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$.

  (ii) Cartesian equation of a line through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

- Angle 'θ' between lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}$.

- Angle θ between lines $\dfrac{x - x_1}{a_1} = \dfrac{y + y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y + y_2}{b_2} = \dfrac{z - z_2}{c_2}$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

- Two lines are perpendicular to each other if

  $\vec{b_1} \cdot \vec{b_2} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

- Equation of plane :

  (i) At a distance of p unit from origin and perpendicular to $\hat{n}$ is $\vec{r} \cdot \hat{n} = p$ and corresponding Cartesian form is $lx + my + nz = p$ when l, m and n are *d.c.*s of normal to plane.

  (ii) Passing through $\vec{a}$ and normal to $\vec{n}$ is $\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ and corresponding Cartesian form is $a(x - x_1) + b\,(y - y_1) + c(z - z_1) = 0$ where a, b, c are *d.r.*'s of normal to plane and (x_1, y_1, z_1) lies on the plane.

  (iii) Passing through three non collinear points is

  $\left(\vec{r} - \vec{a}\right)\cdot\left[\left(\vec{b} - \vec{a}\right)\times\left(\vec{c} - \vec{a}\right)\right] = 0$

or $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

(iv) Having intercepts a, b and c on co-ordinate axis is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

(v) Planes passing through the line of intersection of planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is

$$\left(\vec{r} \cdot \vec{n_1} - d_1\right) + \lambda\left(\vec{r} \cdot \vec{n_2} - d_2\right) = 0.$$

- (i) Angle 'θ' between planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is given by $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}.$

(ii) Angle θ between $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(iii) Two planes are perpendicular to each other iff $\vec{n_1} \cdot \vec{n_2} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(iv) Two planes are parallel iff $\vec{n_1} = \lambda\vec{n_2}$ for some scaler $\lambda \neq 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$

- (i) Distance of a point $\left(\vec{a}\right)$ from plane $\left(\vec{r} \cdot \vec{n} = d\right)$ is $\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}.$

(ii) Distance of a point (x_1, y_1, z_1) from plane $ax + by + cz = d$ is $\frac{|ax_1 + by_1 + cz_1 - d|}{|a^2 + b^2 + c^2|}.$

12. (i) Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are coplanar. Iff $\left(\vec{a_2} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right) = 0$ and equation of plane, containing these lines is $\left(\vec{r} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right) = 0.$

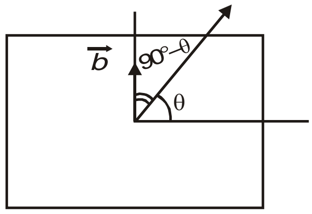
(ii) Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar Iff

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

and equation of plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

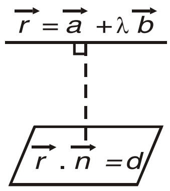
- (i) The angle θ between line $\vec{r} = \vec{a} + \lambda \vec{b}$ and plane $\vec{r} \cdot \vec{n} = d$ is given as $\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$.



(ii) The angle θ between line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and plane $a_2x + b_2y + c_2 z = d$ is given as

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(iii) A line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to plane $\vec{r} \cdot \vec{n} = d$ $\Leftrightarrow \vec{b} \cdot \vec{n} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.



VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. What is the distance of point (a, b, c) from x-axis?
2. What is the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$?
3. Write the equation of a line passing through (2, –3, 5) and parallel to line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$.
4. Write the equation of a line through (1, 2, 3) and perpendicular to $\vec{r} \cdot (\hat{i} - \hat{j} + 3\hat{k}) = 5$.
5. What is the value of λ for which the lines $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$ are perpendicular to each other.
6. If a line makes angle α, β, and γ with co-ordinate axes, then what is the value of

   $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$?
7. Write line $\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{j} - \hat{k})$ into Cartesian form.
8. If the direction ratios of a line are 1, –2, 2 then what are the direction cosines of the line?
9. Find the angle between the planes $2x - 3y + 6z = 9$ and xy – plane.
10. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.
11. What is the perpendicular distance of plane $2x - y + 3z = 10$ from origin?
12. What is the y-intercept of the plane $x - 5y + 7z = 10$?
13. What is the distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$.
14. What is the equation of the plane which cuts off equal intercepts of unit length on the coordinate axes.
15. Are the planes $x + y - 2z + 4 = 0$ and $3x + 3y - 6z + 5 = 0$ intersecting?
16. What is the equation of the plane through the point (1, 4, – 2) and parallel to the plane $-2x + y - 3z = 7$?

17. Write the vector equation of the plane which is at a distance of 8 units from the origin and is normal to the vector $\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$.

18. What is equation of the plane if the foot of perpendicular from origin to this plane is (2, 3, 4)?

19. Find the angles between the planes $\vec{r} \cdot \left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1$ and $\vec{r} \cdot \left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = 0$.

20. What is the angle between the line $\dfrac{x+1}{3} = \dfrac{2y-1}{4} = \dfrac{2-z}{-4}$ and the plane $2x + y - 2z + 4 = 0$?

21. If O is origin $OP = 3$ with direction ratios proportional to –1, 2, – 2 then what are the coordinates of P?

22. What is the distance between the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda\left(\hat{i} + \hat{j} + 4\hat{k}\right)$ from the plane $\vec{r} \cdot \left(-\hat{i} + 5\hat{j} - \hat{k}\right) + 5 = 0$.

23. Write the line $2x = 3y = 4z$ in vector form.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

24. The line $\dfrac{x-4}{1} = \dfrac{2y-4}{2} = \dfrac{k-z}{-2}$ lies exactly in the plane $2x - 4y + z = 7$. Find the value of k.

25. Find the equation of a plane containing the points (0, –1, –1), (–4, 4, 4) and (4, 5, 1). Also show that (3, 9, 4) lies on that plane.

26. Find the equation of the plane which is perpendicular to the plane $\vec{r} \cdot \left(5\hat{i} + 3\hat{j} + 6\hat{k}\right) + 8 = 0$ & which is containing the line of intersection of the planes $\vec{r} \cdot \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 4$ and $\vec{r} \cdot \left(2\hat{i} + \hat{j} - \hat{k}\right) + 5 = 0$.

27. If l_1, m_1, n_1, and l_2, m_2, n_2 are direction cosines of two mutually perpendicular lines, show that the direction cosines of line perpendicular to both of them are

$$m_1n_2 - n_1m_2,\ n_1l_2 - l_1n_2,\ l_1m_2 - m_1l_2.$$

28. Find vector and Cartesian equation of a line passing through a point with position vectors $2\hat{i} + \hat{j} + \hat{k}$ and which is parallel to the line joining the points with position vectors $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$.

29. Find the equation of the plane passing through the point (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x - 5y = 15$.

30. Find equation of plane through line of intersection of planes $\vec{r} \cdot (2\hat{i} + 6\hat{j}) + 12 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$ which is at a unit distance from origin.

31. Find the image of the point (3, –2, 1) in the plane $3x - y + 4z = 2$.

32. Find the equation of a line passing through (2, 0, 5) and which is parallel to line $6x - 2 = 3y + 1 = 2z - 2$.

33. Find image (reflection) of the point (7, 4, – 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

34. Find equations of a plane passing through the points (2, –1, 0) and (3, –4, 5) and parallel to the line $2x = 3y = 4z$.

35. Find distance of the point (– 1, – 5, – 10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

36. Find equation of the plane passing through the points (2, 3, – 4) and (1, –1, 3) and parallel to the x–axis.

37. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$, measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

38. Find the equation of the plane passing through the intersection of two plane $3x - 4y + 5z = 10$, $2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

39. Find the distance between the planes $2x + 3y - 4z + 5 = 0$ and $\vec{r}.(4\hat{i} + 6\hat{j} - 8\hat{k}) = 11$.

40. Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ whose perpendicular distance from the point (1, 2, 3) is 1 unit.

41. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect each other. Find the point of intersection.

42. Find the shortest distance between the lines

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) \text{ and}$$

$$\vec{r} = \left(2\hat{i} + 4\hat{j} + 5\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right).$$

43. Find the distance of the point (–2, 3, –4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

44. Find the equation of plane passing through the point (–1, –1, 2) and perpendicular to each of the plane

$$\vec{r} \cdot \left(2\hat{i} + 3\hat{j} - 3\hat{k}\right) = 2 \text{ and } \vec{r} \cdot \left(5\hat{i} - 4\hat{j} + \hat{k}\right) = 6.$$

45. Find the equation of a plane passing through (–1, 3, 2) and parallel to each of the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$.

46. Show that the plane $\vec{r} \cdot \left(\hat{i} - 3\hat{j} + 5\hat{k}\right) = 7$ contains the line $\vec{r} = \left(\hat{i} + 3\hat{j} + 3\hat{k}\right) + \lambda\left(3\hat{i} + \hat{j}\right)$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

47. Check the coplanarity of lines

$$\vec{r} = \left(-3\hat{i} + \hat{j} + 5\hat{k}\right) + \lambda\left(-3\hat{i} + \hat{j} + 5\hat{k}\right)$$

$$\vec{r} = \left(-\hat{i} + 2\hat{j} + 5\hat{k}\right) + \mu\left(-\hat{i} + 2\hat{j} + 5\hat{k}\right)$$

If they are coplanar, find equation of the plane containing the lines.

48. Find shortest distance between the lines :

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

49. Find shortest distance between the lines :

$$\vec{r} = (1-\lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$$

$$\vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} + (2\mu + 1)\hat{k}.$$

50. A variable plane is at a constant distance $3p$ from the origin and meets the coordinate axes in A, B and C. If the centroid of ΔABC is (α, β, γ), then show that $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = p^{-2}$.

51. A vector $\vec{n}$ of magnitude 8 units is inclined to x–axis at 45°, y axis at 60° and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$, find its equation in vector form.

52. Find the foot of perpendicular from the point $2\hat{i} - \hat{j} + 5\hat{k}$ on the line $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$. Also find the length of the perpendicular.

53. A line makes angles α, β, λ, δ with the four diagonals of a cube. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

54. Find the equation of the plane passing through the intersection of planes $2x + 3y - z = -1$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z = 4$. Also find the inclination of this plane with xy-plane.

ANSWERS

1. $\sqrt{b^2 + c^2}$

2. 90°

3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$.

4. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 3\hat{k})$

5. $\lambda = 2$

6. 2

7. $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$.

8. $\pm\frac{1}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}$

9. $\cos^{-1}(6/7)$.

10. $\dfrac{x}{a} = \dfrac{y-1}{a} = \dfrac{z-2}{a}$, $a \in R - \{0\}$

11. $\dfrac{10}{\sqrt{14}}$

12. -2

13. $\dfrac{1}{6}$

14. $x + y + z = 1$

15. No

16. $-2x + y - 3z = 8$

17. $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24$

18. $2x + 3y + 4z = 29$

19. $\cos^{-1}\left(\dfrac{11}{21}\right)$

20. 0 (line is parallel to plane)

21. $(-1, 2, -2)$

22. $\dfrac{10}{3\sqrt{3}}$

23. $\vec{r} = \vec{o} + \lambda(6\hat{i} + 4\hat{j} + 3\hat{k})$

24. $k = 7$

25. $5x - 7y + 11z + 4 = 0$.

26. $\vec{r} \cdot (-51\hat{i} - 15\hat{j} + 50\hat{k}) = 173$

28. $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\dfrac{x-2}{2} = \dfrac{y+1}{-2} = \dfrac{z-1}{1}$.

29. $x - 2y + 3z = 1$

30. $\vec{r} \cdot (8\hat{i} + 4\hat{j} + 8\hat{k}) + 12 = 0$ or $\vec{r} \cdot (-4\hat{i} + 8\hat{j} - 8\hat{k}) + 12 = 0$

31. $(0, -1, -3)$

32. $\dfrac{x-2}{1} = \dfrac{y}{2} = \dfrac{z-5}{3}$.

33. $\left(\dfrac{47}{7}, -\dfrac{18}{7}, \dfrac{43}{7}\right)$

34. $29x - 27y - 22z = 85$

35. 13

36. $7y + 4z = 5$

37. 1

38. $x - 20y + 27z = 14$

39. $\frac{21}{2\sqrt{29}}$ units.

40. $x - 2y + 2z = 0$ or $x - 2y + 2z = 6$

41. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

42. $\frac{1}{\sqrt{6}}$

43. $\frac{17}{2}$

44. $\vec{r} \cdot \left(9\hat{i} + 17\hat{j} + 23\hat{k}\right) = 20$

45. $2x - 7y + 4z + 15 = 0$

47. $x - 2y + z = 0$

48. $\frac{16}{7}$

49. $\frac{8}{\sqrt{29}}$

51. $\vec{r} \cdot \left(\sqrt{2}\hat{i} + \hat{j} + \hat{k}\right) = 2$

52. $\hat{i} + 2\hat{j} + 3\hat{k}, \sqrt{14}$

54. $7x + 13y + 4z = 9, \cos^{-1}\left(\frac{4}{\sqrt{234}}\right)$.