CHAPTER 10

VECTORS

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a *vector*. It is denoted by a directed line segment.
- Two or more vectors which are parallel to same line are called *collinear vectors*.
- *Position vector* of a point *P*(*a, b, c*) w.r.t. origin (0, 0, 0) is denoted by $\overrightarrow{OP}$, where $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$.
- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of triangle taken in opposite order. This is called triangle *law of addition of vectors.*
- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda\vec{a}$ is a vector collinear with $\vec{a}$ and $|\lambda\vec{a}| = |\lambda||\vec{a}|$.
- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\vec{b}$ where λ is some scalar.
- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$, where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points *A* and *B,* and *C* is any point which divides $\overrightarrow{AB}$ in ratio *m* : *n* internally then position vector $\vec{c}$ of point *C* is given as $\vec{C} = \dfrac{m\vec{b} + n\vec{a}}{m + n}$. If *C* divides $\overrightarrow{AB}$ in ratio *m* : *n* externally, then $\vec{C} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$.

- The angles α, β and γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, *y* and *z*-axis are called direction angles and cosines of these angles are called *direction cosines* of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

  Also $l = \dfrac{a}{|\vec{r}|}$, $m = \dfrac{b}{|\vec{r}|}$, $n = \dfrac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$.

- The numbers *a, b, c* proportional to *l, m, n* are called *direction ratios.*

- *Scalar product* of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}.\vec{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.

- *Dot product* of two vectors is commutative *i.e.* $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{o}, \vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, so $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1a_2 + b_1b_2 + c_1c_2$.

- *Projection of* $\vec{a}$ on $\vec{b} = \left|\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}\right|$ and projection vector of

$$\vec{a} \text{ along } \vec{b} = \left(\frac{(\vec{a}\,.\,\vec{b})}{|\vec{b}|}\right)\hat{b}.$$

- *Cross product* or vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$. were θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ *and* $\hat{n}$ form a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$.
- $\vec{a} \times \vec{b} = \vec{o} \Leftrightarrow \vec{a} = \vec{o},\ \vec{b} = \vec{o}$ or $\vec{a} \parallel \vec{b}$.
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{o}$.
- $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\left(\dfrac{\left(\vec{a} \times \vec{b}\right)}{\left|\vec{a} \times \vec{b}\right|}\right)$.
- $\left|\vec{a} \times \vec{b}\right|$ is the *area of parallelogram* whose adjacent sides are $\vec{a}$ and $\vec{b}$.
- $\dfrac{1}{2}\left|\vec{a} \times \vec{b}\right|$ is the area of parallelogram where diagonals are $\vec{a}$ and $\vec{b}$.
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ forms a triangle, then *area of the triangle.*

$$= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|\vec{b} \times \vec{c}\right| = \frac{1}{2}\left|\vec{c} \times \vec{a}\right|.$$

- Scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ and is denoted as $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

- Geometrically, absolute value of scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents volume of a parallelepiped whose coterminous edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$.

- $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

- $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ &
  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The scalar triple product of three vectors is zero if any two of them are same or collinear.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. What are the horizontal and vertical components of a vector $\vec{a}$ of magnitude 5 making an angle of 150° with the direction of x-axis.

2. What is $a \in R$ such that $|a\vec{x}| = 1$, where $\vec{x} = \hat{i} - 2\hat{j} + 2\hat{k}$?

3. When is $|\vec{x} + \vec{y}| = |\vec{x}| + |\vec{y}|$?

4. What is the area of a parallelogram whose sides are given by $2\hat{i} - \hat{j}$ and $\hat{i} + 5\hat{k}$?

5. What is the angle between $\vec{a}$ *and* $\vec{b}$, If
   $\vec{a} \cdot \vec{b} = 3$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$.

6. Write a unit vector which makes an angle of $\frac{\pi}{4}$ with x-axis and $\frac{\pi}{3}$ with z-axis and an acute angle with y-axis.

7. If A is the point (4, 5) and vector $\overrightarrow{AB}$ has components 2 and 6 along x-axis and y-axis respectively then write point B.

8. What is the point of trisection of PQ nearer to P if positions of P and Q are $3\hat{i} + 3\hat{j} - 4\hat{k}$ and $9\hat{i} + 8\hat{j} - 10\hat{k}$ respectively?

9. Write the vector in the direction of $2\hat{i} + 3\hat{j} + 2\sqrt{3}\hat{k}$, whose magnitude is 10 units.

10. What are the direction cosines of a vector equiangular with co-ordinate axes?

11. What is the angle which the vector $3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the x-axis?

12. Write a unit vector perpendicular to both the vectors

$$3\hat{i} - 2\hat{j} + \hat{k} \text{ and } -2\hat{i} + \hat{j} - 2\hat{k}.$$

13. What is the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$?

14. If $|\vec{a}| = 2$, $|\vec{b}| = 2\sqrt{3}$ and $\vec{a} \perp \vec{b}$, what is the value of $|\vec{a} + \vec{b}|$?

15. For what value of λ, $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ is perpendicular to $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$?

16. What is $|\vec{a}|$, if $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 3$ and $2|\vec{b}| = |\vec{a}|$?

17. What is the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}|$?

18. In a parallelogram $ABCD$, $\overrightarrow{AB} = 2\hat{i} - \hat{j} + 4\hat{k}$ *and* $\overrightarrow{AC} = \hat{i} + \hat{j} + 4\hat{k}$. What is the length of side BC?

19. What is the area of a parallelogram whose diagonals are given by vectors $2\hat{i} + \hat{j} - 2\hat{k}$ and $-\hat{i} + 2\hat{k}$?

20. Find $|\vec{x}|$ if for a unit vector $\hat{a}$, $(\vec{x} - \hat{a}).(\vec{x} + \hat{a}) = 12$.

21. If $\vec{a}$ and $\vec{b}$ are two unit vectors and $\vec{a} + \vec{b}$ is also a unit vector then what is the angle between $\vec{a}$ and $\vec{b}$?

22. If $\hat{i}, \hat{j}, \hat{k}$ are the usual three mutually perpendicular unit vectors then what is the value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{j} \times \hat{i})$?

23. What is the angle between $\vec{x}$ and $\vec{y}$ if $\vec{x} . \vec{y} = |\vec{x} \times \vec{y}|$?

24. Write a unit vector in *xy*-plane, making an angle of 30° with the +ve direction of *x*–axis.

25. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors with $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then what is the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$?

26. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\left(\vec{a} + 2\vec{b}\right)$ is perpendicular to $\left(5\vec{a} - 4\vec{b}\right)$, then what is the angle between $\vec{a}$ and $\vec{b}$?

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

27. If ABCDEF is a regular hexagon then using triangle law of addition prove that :

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD} = 6\overrightarrow{AO}$$

O being the centre of hexagon.

28. Points *L, M, N* divides the sides *BC, CA, AB* of a ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$ where *K* divides *AB* in ratio 1 : 3.

29. The scalar product of vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ *and* $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ.

30. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude. Show that $\vec{a} + \vec{b} + \vec{c}$ makes equal angles with $\vec{a}$, $\vec{b}$ and $\vec{c}$ with each angle as $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

31. If $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

32. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

33. If $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the angle between $\vec{a}$ and $\vec{b}$.

34. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$, find a vector $\vec{d}$ which is perpendicular to $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 1$.

35. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are the given vectors then find a vector $\vec{b}$ satisfying the equation $\vec{a} \times \vec{b} = \vec{c}$, $\vec{a} \cdot \vec{b} = 3$.

36. Find a unit vector perpendicular to plane *ABC,* when position vectors of *A, B, C* are $3\hat{i} - \hat{j} + 2\hat{k}$, $\hat{i} - \hat{j} - 3\hat{k}$ and $4\hat{i} - 3\hat{j} + \hat{k}$ respectively.

37. For any two vector, show that $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

38. Evaluate $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$.

39. If $\hat{a}$ and $\hat{b}$ are unit vector inclined at an angle θ than prove that :

    (i) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$. (ii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} + \hat{b}}\right|$.

40. For any two vectors, show that $|\vec{a} \times \vec{b}| = \sqrt{a^2b^2 - (\vec{a} \cdot \vec{b})^2}$.

41. $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{j}$ and $\vec{c} = x\hat{i} + (x - 2)\hat{j} - \hat{k}$. If $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then find the value of *x*.

42. Prove that angle between any two diagonals of *a* cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

43. Let $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors such that $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ is $\frac{\pi}{6}$, then prove that $\hat{a} = \pm 2(\hat{b} \times \hat{c})$.

44. Prove that the normal vector to the plane containing three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lies in the direction of vector $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$.

45. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices A, B, C of a triangle ABC then show that the area of ΔABC is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$.

46. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then prove that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$ provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

47. Dot product of a vector with vectors $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ is 0, 5 and 8 respectively. Find the vectors.

48. If $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$, $\hat{b} = \hat{i} - \hat{j} - \lambda\hat{k}$, find λ such that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

49. Let $\vec{a}$ and $\vec{b}$ be vectors such that $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{a} - \vec{b}\right| = 1$, then find $\left|\vec{a} + \vec{b}\right|$.

50. If $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 5$ and $\vec{a} \times \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the value of $\vec{a} \cdot \vec{b}$.

51. $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{b} \times \vec{c} = \vec{a}$ and $\vec{a} \times \vec{b} = \vec{c}$. Prove that $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular to each other and $\left|\vec{b}\right| = 1$, $\left|\vec{c}\right| = \left|\vec{a}\right|$.

52. If $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ *and* $\vec{c} = 3\hat{i} - \hat{k}$ find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

53. Find volume of parallelepiped whose coterminous edges are given by vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.

54. Find the value of λ such that $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar.

55. Show that the four points (–1, 4, –3), (3, 2, –5) (–3, 8, –5) and (–3, 2, 1) are coplanar.

56. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, prove that

$$\left[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}\right] = 2\left[\vec{a} \ \vec{b} \ \vec{c}\right]$$

57. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, prove that $\vec{a} - \vec{b}$, $\vec{b} - \vec{c}$ and $\vec{c} - \vec{a}$ are coplanar.

ANSWERS

1. $-\frac{5\sqrt{3}}{2}, \frac{5}{2}$.
2. $a = \pm\frac{1}{3}$
3. $\vec{x}$ and $\vec{y}$ are like parallel vectors.
4. $\sqrt{126}$ sq units.
5. $\frac{\pi}{3}$
6. $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$
7. (6, 11)
8. $\left(5, \frac{14}{3}, -6\right)$
9. $4\hat{i} + 6\hat{j} + 4\sqrt{3}\hat{k}$.
10. $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.
11. $\cos^{-1}\left(\frac{3}{7}\right)$.
12. $\frac{3\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{26}}$.
13. 0
14. 4
15. –9
16. 2
17. $\frac{\pi}{2}$.
18. $\sqrt{5}$
19. $\frac{3}{2}$ sq. units.

20. $\sqrt{13}$

21. $\frac{2\pi}{3}$

22. -1

23. $\frac{\pi}{4}$

24. $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

25. $-\frac{3}{2}$

26. $\frac{\pi}{3}$

29. $\lambda = 1$

31. $\vec{\beta} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$.

33. 60°

34. $\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{3}{4}\hat{k}$.

35. $\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$.

36. $\frac{-1}{\sqrt{165}}\left(10\hat{i} + 7\hat{j} - 4\hat{k}\right)$.

38. $2|\vec{a}|^2$

41. $x = -2$

47. $\hat{i} + 2\hat{j} + \hat{k}$

48. $\pm\sqrt{73}$

49. $\sqrt{3}$

50. $\frac{91}{10}$

52. 4

53. 37

54. $\lambda = 1$